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12068.Proposed by D. M. Bătinetu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu,"George Emil Palade" School, Buzău, Romania.

Consider a triangle with altitudes h_a,h_b, and h_c and corresponding exradii r_a,r_b, and r_c. Let s, r, and R denote the triangle's semiperimeter, inradius, and circumradius, respectively.

(**a**) Prove $\sum \frac{h_b+h_c}{h_a}\cdot r_a^2 \geq 2s^2$.

(**b**) Prove $\sum \frac{r_b+r_c}{r_a}\cdot h_a^2 \geq \frac{4s^2r}{R}$.

Solution by Arkady Alt, San Jose, California, USA.

Let F be area of the triangle. Since $h_t = \frac{2F}{t}$, $r_t = \frac{F}{s-t} = \frac{s\cdot r}{s-t}$,
where $t \in \{a,b,c\}$ and $r^2 = s(s-a)(s-b)(s-c)$ then

$\sum \frac{h_b+h_c}{h_a}\cdot r_a^2 \geq 2s^2 \Leftrightarrow \sum \frac{ab+ca}{bc}\cdot\frac{s^2\cdot r^2}{(s-a)^2} \geq 2s^2 \Leftrightarrow$

$\sum \frac{ab+ca}{bc(s-a)^2} \geq \frac{2}{r^2} \Leftrightarrow \sum a^2(b+c)(s-b)^2(s-c)^2 \geq$

$\frac{2abc(s-a)^2(s-b)^2(s-c)^2}{r^2} \Leftrightarrow$

(**1**) $\sum a^2(b+c)(s-b)^2(s-c)^2 \geq 2abc\cdot r^2\cdot s^2$.

Also since $\frac{r_b+r_c}{r_a} = \frac{(2s-(b+c))(s-a)}{(s-c)(s-b)} = \frac{a(s-a)}{(s-c)(s-b)}$ and $h_a^2 = \frac{4s^2r^2}{a^2}$

then $\sum \frac{r_b+r_c}{r_a}\cdot h_a^2 \geq \frac{4s^2r}{R} \Leftrightarrow \sum \frac{a(s-a)}{(s-c)(s-b)}\cdot\frac{4s^2r^2}{a^2} \geq \frac{4s^2r}{R} \Leftrightarrow$

$\sum \frac{s-a}{a(s-c)(s-b)} \geq \frac{1}{R\cdot r} \Leftrightarrow \sum bc(s-a)^2 \geq \frac{abc(s-a)(s-b)(s-c)}{Rr} \Leftrightarrow$

(**2**) $\sum bc(s-a)^2 \geq 4r^2s^2$.

Let $x := s-a, y := s-b, z := s-c, p := xy+yz+zx, q := xyz.$

Then, assuming $s = 1$ (due homogeneity) we obtain $x,y,z > 0$,

$x+y+z = 1, a = 1-x, b = 1-y,\ c = 1-z, abc = p-q$,

$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{q}, R = \frac{abc}{4rs} = \frac{p-q}{4\sqrt{q}}$,

$\sum a^2(b+c)(s-b)^2(s-c)^2 = \sum(1-x)^2(1+x)y^2z^2 =$
$\sum(x^3y^2z^2 - x^2y^2z^2 - xy^2z^2 + y^2z^2) = q^2-3q^2-pq+p^2-2q =$
$p^2-pq-2q^2-2q$ and, therefore,

$\sum a^2(b+c)(s-b)^2(s-c)^2 - 2abc\cdot r^2\cdot s^2 =$
$q^2-3q^2-pq+p^2-2q-2(p-q)q = p^2 - q(3p+2)$

Since $3q = 3xyz(x+y+z) \leq (xy+yz+zx)^2 = p^2$ and
$3p = 3(xy+yz+zx) \leq (x+y+z)^2 = 1$ then

$p^2 - q(3p+2) \geq p^2 - \frac{p^2}{3}(3p+2) = \frac{p^2(1-3p)}{3} \geq 0.$

Also, since $\sum bc(s-a)^2 = \sum(1-y)(1-z)x^2 = \sum(x+yz)x^2 =$

$\sum(x^3 + x^2yz) = (1 + 3q - 3p) + q = 4q - 3p + 1$ then
$\sum bc(s - a)^2 - 4r^2s^2 = 4q - 3p + 1 - 4q = 1 - 3p \geq 0.$